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DERIVATION OF LOGARITHMIC SOBOLEV TRACE INEQUALITIES

Young Ja Park

Abstract. Logarithmic Sobolev trace inequalities are derived from the well known classical Sobolev trace inequalities as a limiting case.

1. Introduction

The classical Sobolev trace inequalities are written as:

$$\left(\int_{\mathbb{R}^n} |f(x)|^t dx\right)^{s/t} \leq A_{s,t} \int_{\mathbb{R}^{n+1}_+} |\nabla u(x,y)|^s dxdy \tag{1.1}$$

with

$$\frac{1}{t} = \frac{n+1}{ns} - \frac{1}{n},$$

where u is the extension of f to the upper half-space and $A_{s,t}$ is a positive constant independent of u. Many mathematicians have developed this type of inequalities using various methods in different settings ([3], [5]). These inequalities of Sobolev type provide estimates of lower order derivatives of the trace function f in terms of higher order derivatives of u.

Logarithmic Sobolev trace inequalities capture the spirit of classical Sobolev trace inequalities with the logarithm function replacing powers, and they can be considered as limiting cases of Sobolev trace inequalities. Recently a logarithmic trace inequality in a gaussian space was introduced in [4]. It states: for a smooth domain Ω and $u \in C^\infty(\bar{\Omega})$,

$$\frac{1}{\sqrt{2\pi}^{\,n}} \int_{\partial\Omega} |u|^p \left\{\ln(2+|u|)\right\}^{\frac{p-1}{2}} e^{\frac{-|x|^2}{2}} dS \leq C\|u\|^p_{W^{1,p}(\Omega,\gamma)},$$

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where C is an absolute constant. Using this, the trace operator in the weighted Sobolev space for sufficiently regular domain was investigated, and the results were applied to the theory of partial differential equations and the quantum field theory. We refer [4] for details.

Another special version of logarithmic Sobolev trace inequalities was investigated in [9]: for $f \in \mathcal{S}(\mathbb{R}^n)$ with $\|f\|_{L^2(\mathbb{R}^n)} = 1$ and $n > 1$,

$$(1.2)\quad \left(\int_{\mathbb{R}^n} |f(x)|^2 \ln |f(x)| dx\right) \leq \frac{n}{2} \ln \left(A_n \int_{\mathbb{R}^{n+1}_+} |\nabla u(x,y)|^2 dxdy\right),$$

where u is an extension of f to the upper half-space that is continuous in the closed upper half-space and at least once differentiable on the open upper half-space, and A_n is a positive constant dependent only on the dimension n. The logarithmic uncertainty principle was utilized to derive the logarithmic inequality (1.2).

This paper derives a general logarithmic Sobolev trace inequality which is a generalization of the logarithmic trace inequality (1.2) and is a limiting case of the general classical Sobolev trace inequalities. It states:

THEOREM 1. *Let $n > 1$. For any measurable function u satisfying $\nabla u \in L^p(\mathbb{R}^{n+1}_+)$ and*

$$\|f\|_{L^q(\mathbb{R}^n)} = 1$$

with $u(x,0) = f(x)$ in the sense of distribution, we have

$$(1.3)\quad \left(\int_{\mathbb{R}^n} |f(x)|^q \ln |f(x)| dx\right) \leq \frac{n}{p} \ln \left(A_{p,\frac{nq}{n-1}} \int_{\mathbb{R}^{n+1}_+} |\nabla u(x,y)|^p dxdy\right)$$

under the conjugate condition

$$\frac{n+1}{p} = \frac{n-1}{q} + 1.$$

The constant $A_{p,\frac{nq}{n-1}}$ is the same positive constant independent of u appeared in (1.1).

We point out that the inequality (1.3) with $p = 2$ is reduced to the inequality (1.2).

The proof of the theorem 1 relies on the following lemma.

LEMMA 2. [7] *Assume* $f \in L^{p_0}(X,\mu)$ *for some* $0 < p_0 \leq \infty$ *and* $\mu(X) = 1$. *Then we have* $f \in L^p(X,\mu)$ *for* $0 < p \leq p_0$, *and that*

$$\lim_{p\to 0}\left(\int_X |f|^p d\mu\right)^{\frac{1}{p}} = \exp\left(\int_X \ln|f| d\mu\right).$$

2. The argument

We assume that f belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ of functions on $\mathbb{R}^n$. From the fact that $\|f\|_{L^q(\mathbb{R}^n)} = 1$, Lemma 2 with respect to the probability measure $|f(x)|^q dx$ yields

$$\begin{aligned}\lim_{r\to q}\left(\int_{\mathbb{R}^n}|f(x)|^r dx\right)^{\frac{1}{r-q}} &= \lim_{r\to q}\left(\int_{\mathbb{R}^n}|f(x)|^{r-q}|f(x)|^q dx\right)^{1/(r-q)}\\ &= \exp\left(\int_{\mathbb{R}^n}|f(x)|^q \ln|f(x)| dx\right).\end{aligned}$$

On the other hand, we split the index r into two numbers, and apply Hölder's inequality to get:

$$\begin{aligned}\int_{\mathbb{R}^n}|f(x)|^r dx &= \int_{\mathbb{R}^n}|f(x)|^{n(r-q)+\{r-n(r-q)\}} dx\\ &\leq \left(\int_{\mathbb{R}^n}|f(x)|^{n(r-q)\alpha}dx\right)^{\frac{1}{\alpha}}\left(\int_{\mathbb{R}^n}|f(x)|^{\{r-n(r-q)\}\beta}dx\right)^{\frac{1}{\beta}},\end{aligned}$$

where $1 < \alpha < \infty$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Here β can be chosen to satisfy that

$$\{r - n(r-q)\}\beta = q.$$

Then the assumption $\|f\|_{L^q(\mathbb{R}^n)} = 1$ can be used to simplify the inequality as follows:

$$\begin{aligned}\left(\int_{\mathbb{R}^n}|f(x)|^r dx\right)^{\frac{1}{r-q}} &\leq \left(\int_{\mathbb{R}^n}|f(x)|^{n(r-q)\alpha}dx\right)^{\frac{1}{\alpha(r-q)}}\\ &= \left(\int_{\mathbb{R}^n}|f(x)|^{\frac{nq}{n-1}}dx\right)^{\frac{n-1}{q}}.\end{aligned}$$

Apply the Sobolev trace inequality to achieve that for $f \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}\left(\int_{\mathbb{R}^n}|f(x)|^r dx\right)^{\frac{1}{r-q}} &\leq \left(\int_{\mathbb{R}^n}|f(x)|^{\frac{nq}{n-1}}dx\right)^{\frac{n-1}{q}}\\ &\leq \left(A_{p,\frac{nq}{n-1}}\int_{\mathbb{R}^{n+1}_+}|\nabla u(x,y)|^p dxdy\right)^{\frac{n}{p}},\end{aligned}$$

where the indices p and q satisfy

$$\frac{n+1}{p} = \frac{n-1}{q} + 1.$$

We now take the limit on both sides of the above inequality to get

$$\exp\left(\int_{\mathbb{R}^n}|f(x)|^q \ln|f(x)|dx\right) \leq \left(A_{p,\frac{nq}{n-1}}\int_{\mathbb{R}^{n+1}_+}|\nabla u(x,y)|^p dxdy\right)^{\frac{n}{p}}.$$

For $f \in \mathcal{S}(\mathbb{R}^n)$ with $\|f\|_{L^q(\mathbb{R}^n)} = 1$, we have the generalized logarithmic Sobolev trace inequality of the form:

$$\left(\int_{\mathbb{R}^n}|f(x)|^q \ln|f(x)|dx\right) \leq \frac{n}{p}\ln\left(A_{p,\frac{nq}{n-1}}\int_{\mathbb{R}^{n+1}_+}|\nabla u(x,y)|^p dxdy\right).$$

The density argument completes the proof.

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Department of Mathematics
Hoseo University
Hoseo-ro 79, Asan, Chungnam 31499
Republic of Korea
E-mail: ypark@hoseo.edu